THE EISENSTEIN ELEMENTS OF MODULAR SYMBOLS FOR LEVEL PRODUCT OF TWO DISTINCT ODD PRIMES

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Abstract. We explicitly write down the *Eisenstein elements* inside the space of modular symbols for Eisenstein series with integer coefficients for the congruence subgroups $\Gamma_0(pq)$ with p and q distinct odd primes, giving an answer to a question of Merel in these cases. We also compute the winding elements explicitly for these congruence subgroups. Our results are explicit versions of the Manin-Drinfeld Theorem [Thm. 9].

1. Introduction

In his landmark paper on Eisenstein ideals, Mazur studied torsion points of elliptic curves over $\mathbb{Q}$ and gave a list of possible torsion subgroups of elliptic curves [cf. Thm. 8, [9]]. In [14], Merel wrote down modular symbols for the congruence subgroups $\Gamma_0(p)$ for any odd prime p that correspond to differential forms of third kind on the modular curves. He then use these modular symbols to give an uniform upper bound of the torsion points of elliptic curves over any number fields in terms of extension degrees of these number fields [13]. The explicit expressions of winding elements for prime level of [14] are being used by Calegari and Emerton to study the ramifications of Hecke algebras at the Eisenstein primes [3]. Several authors afterwards studied the torsion points of elliptic curves over number fields using modular symbols.

In the present paper, we study elements of relative homology groups of the modular curve $X_0(pq)$ that correspond to differential forms of third kind with p and q distinct odd primes. As a consequence, we give an "effective" proof of the Manin-Drinfeld theorem [Thm. 9] for the special case of the image in $\mathrm{H}_1(X_0(pq),\mathbb{R})$ of the path in $\mathrm{H}_1(X_0(pq),\partial(X_0(pq)),\mathbb{Z})$ joining 0 and $i\infty$. Since the algebraic part of the special values of L-function are obtained by integrating differential forms on these modular symbols, our explicit expression of the winding elements should be useful to understand the algebraic parts of the special values at 1 of the L-functions of the quotient Jacobian of modular curves for the congruence subgroup $\Gamma_0(pq)$ [1].

For $N \in \{p,q,pq\}$, consider the basis E_N of $E_2(\Gamma_0(pq))$ [§ 4] for which all the Fourier coefficients at $i\infty$ belong to $\mathbb{Z}$. The meromorphic differential forms $E_N(z)dz$ are of third kind on the Riemann surface $X_0(pq)$ but of first kind on the non-compact Riemann surface $Y_0(pq)$.

Let $\xi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{H}_1(X_0(pq),\mathrm{cusps},\mathbb{Z})$ be the Manin map [§ 3]. For any two coprime integers u and v with $v \geq 1$, let $S(u,v) \in \mathbb{Z}$ be the Dedekind sum [cf 4.1]. If $g \in \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$ is not of the form $(\pm 1,1),(\pm 1 \pm kx,1)$ or $(1,\pm 1 \pm kx)$ with x one of the prime p or q, then we can write it as $(r-1,r+1)$.

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Let δ_r be 1 or 0 depending on r is odd or even. For any integer k, let $s_k = (k + (\delta_k - 1)pq)$ be an odd integer. Choose integers s, s' and l, l' such that $l(s_k x + 2) - 2spq = 1$ and $l' s_k x - 2s'\frac{pq}{x} = 1$. Let $\gamma_1^{x,k} = \begin{pmatrix} 1+4spq & -2l \\ -4s(s_k x+2)pq & 1+4spq \end{pmatrix}$ and $\gamma_2^{x,k} = \begin{pmatrix} 1+4s'\frac{pq}{x} & -2l' \\ -4s'(s_k)pq & 1+4s'\frac{pq}{x} \end{pmatrix}$ be two matrices [cf. Lemma 28]. For $l = 1, 2$, consider the integers

$$P_N(\gamma_l^{x,k}) = sgn(t(\gamma_l^{x,k}))[2(S(s(\gamma_l^{x,k}), |t(\gamma_l^{x,k})|N) - S(s(\gamma_l^{x,k}), |t(\gamma_l^{x,k})|))$$

$$-S(s(\gamma_l^{x,k}), \frac{|t(\gamma_l^{x,k}|)}{2}N) + S(s(\gamma_l^{x,k}), \frac{|t(\gamma_l^{x,k})|}{2})]$$

with

$$s(\gamma_1^{x,k}) = 1 - 4spq(1 + s_k x), t(\gamma_1^{x,k}) = -2(l - 2s(s_k x + 2)pq)$$

and

$$s(\gamma_2^{x,k}) = 1 - 4s'pq(s_k - \frac{1}{x}), t(\gamma_2^{x,k}) = -2(l' - 2s' s_k pq).$$

Define the function $F_N : \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z}) \to \mathbb{Z}$ by

$$F_N(g) = \begin{cases} 2(S(r,N) - 2S(r,2N)) & \text{if } g = (r-1, r+1), \\ P_N(\gamma_1^{x,k}) - P_N(\gamma_2^{x,k}) & \text{if } g = (1+kx, 1) \text{ or } g = (-1-kx, 1), \\ -P_N(\gamma_1^{x,k}) + P_N(\gamma_2^{x,k}) & \text{if } g = (1, 1+kx) \text{ or } g = (1, -1-kx), \\ 0 & \text{if } g = (\pm 1, 1). \end{cases}$$

Theorem 1. *The modular symbol*

$$\mathcal{E}_{E_N} = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} F_{E_N}(g)\xi(g)$$

in $\mathrm{H}_1(X_0(pq), \partial(X_0(pq), \mathbb{Z})$ *is the Eisenstein element [§ 5] corresponding to the Eisenstein series* $E_N \in E_2(\Gamma_0(pq))$.

In [2], a description is given of Eisenstein elements in terms of certain integrals for $M = p^2$. In this article, we give an explicit description in terms of two matrices $\gamma_1^{x,k}$ and $\gamma_2^{x,k}$. Let $\overline{B_1} : \mathbb{R} \to \mathbb{R}$ be the periodic first Bernoulli polynomial. For the Eisenstein series E_{pq} [§ 4], we write down the Eisenstein elements more explicitly if $g = (r-1, r+1)$. Replacing p with pq [Lemma 4, [14]], we write

$$F_{pq}((r-1, r+1)) = \sum_{h=0}^{pq-1} \overline{B_1}(\frac{hr}{2pq}).$$

Recall the concept of the *winding elements* [§ 37]. We write down the explicit expression of the winding elements for the congruence subgroup $\Gamma_0(pq)$.

Corollary 2.

$$(1 - pq)e_{pq} = \sum_{x \in (\mathbb{Z}/pq\mathbb{Z})^*} F_{pq}((1, x))\{0, \frac{1}{x}\}.$$

Note that if $\nu = gcd(pq-1, 12)$ and $n = \frac{pq-1}{\nu}$, then a multiple of winding element ne_{pq} belongs to $\mathrm{H}_1(X_0(pq), \mathbb{Z})$. Manin-Drinfeld proved that the modular symbol $\{0, \infty\} \in \mathrm{H}_1(X_0(N), \mathbb{Q})$ using the theory of suitable Hecke operators acting on modular curve $X_0(N)/\mathbb{Q}$. In this paper, we follow the approach of Merel [cf. [14], Prop. 11]. Our explicit expression of winding element should be useful to understand the algebraic part of the *special values of L-functions* [cf. [1], p. 26].

Since Hecke operators are defined over $\mathbb{Q}$, there is a possibility that we can find the Eisenstein elements for the congruence subgroups of odd level in a completely different method without using boundary computations. It is tempting to remark that our method should generalize to the congruence subgroup $\Gamma_0(N)$ atleast if N is squarefree and odd. Unfortunately, generalizing our method is equivalent to explicit understanding of boundary homologies of modular curves defined over rationals. For instance, if $N = pqr$ with p, q, r three distinct primes then there are 8 cusps. Since in these cases, there are more cusps the computations of boundaries become much more tedious. One of the author wish to tackle the difficulty using the "level" of the cusps in a future article.

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3. Modular Symbols

Let $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) = \overline{\mathbb{H}}$ and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. The topological space $X_\Gamma(\mathbb{C}) = \Gamma \backslash \overline{\mathbb{H}}$ has a natural structure of a smooth compact Riemann surface and consider the usual projection map $\pi : \overline{\mathbb{H}} \to X_\Gamma(\mathbb{C})$. Recall, the map π is unramified outside the elliptic points and the set of cusps $\partial(X_\Gamma)$. Both these sets are finite.

3.1. **The rational structure of the curve $X_0(N)$ defined over rational.** There is a smooth projective curve $X_0(N)$ defined over $\mathbb{Q}$ for which the space $\Gamma_0(N)\backslash\overline{\mathbb{H}}$ is canonically identified with the set of $\mathbb{C}$-points of the projective curve $X_0(N)$. We are interested to understand the $\mathbb{Q}$-structure of the compactified modular curve $X_0(N)$.

3.2. **Classical modular symbols.** Recall the following fundamental theorem of Manin [8].

Theorem 3. *For $\alpha \in \overline{\mathbb{H}}$, consider the map $c : \Gamma \to \mathrm{H}_1(X_0(N), \mathbb{Z})$ defined by*

$$c(g) = \{\alpha, g\alpha\}.$$

The map c is a surjective group homomorphism which does not depend on the choice of point α. The kernel of this homomorphism is generated by

(1) *the commutator,*
(2) *the elliptic elements,*

(3) *the parabolic elements*

of the congruence subgroup Γ.

In particular, the above theorem implies that $\{\alpha, g\alpha\} = 0$ for all $\alpha \in \mathbb{P}^1(\mathbb{Q})$ and $g \in \Gamma$.

3.3. **The Manin map.** Let T, S be the matrices $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $R = ST$ be the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. The modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by S and T.

Theorem 4 (Manin)**.** *[8] Let*

$$\xi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{H}_1(X_0(pq), \partial(X_0(pq)), \mathbb{Z})$$

be the map that takes a matrix $g \in \mathrm{SL}_2(\mathbb{Z})$ *to the class in* $\mathrm{H}_1(X_0(pq), \partial(X_0(pq)), \mathbb{Z})$ *of the image in* $X_0(pq)$ *of the geodesic in* $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ *joining* $g.0$ *and* $g.\infty$.

- *The map* ξ *is surjective.*
- *For all* $g \in \Gamma_0(pq)\backslash\mathrm{SL}_2(\mathbb{Z})$, $\xi(g) + \xi(gS) = 0$ *and* $\xi(g) + \xi(gR) + \xi(gR^2) = 0$.

We have a short exact sequence,

$$0 \to \mathrm{H}_1(X_0(pq), \mathbb{Z}) \to \mathrm{H}_1(X_0(pq), \partial(X_0(pq)), \mathbb{Z}) \to \mathbb{Z}^{\partial(X_0(pq))} \xrightarrow{\delta'} \mathbb{Z} \to 0.$$

The first map is a canonical injection. The boundary map δ' takes a geodesic, joining the cusps r and s to the formal symbol $[r] - [s]$ and the third map is the sum of the coefficients.

3.4. **Relative homology group** $\mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z})$**.** Consider the points $i = \sqrt{-1}$ and $\rho = \frac{1+\sqrt{-3}}{2}$ on the complex upper half plane with ν the geodesic joining i and ρ. These are the elliptic points on the Riemann surface $X_0(pq)$. The projection map π is unramified outside cusps and elliptic points.

Say $R = \pi(\mathrm{SL}_2(\mathbb{Z})\rho)$ and $I = \pi(\mathrm{SL}_2(\mathbb{Z})i)$ be the image of these two sets in $X_0(pq)$. These two sets are disjoint. Consider now the relative homology group $\mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z})$. For $g \in \mathrm{SL}_2(\mathbb{Z})$, let $[g]_*$ be the class of $\pi(g\nu)$ in the relative homology group $\mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z})$. Let $\rho^* = -\overline{\rho}$ be another point on the boundary of the fundamental domain. The homology groups $\mathrm{H}_1(Y_0(pq), \mathbb{Z})$ are subgroups of $\mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z})$. Suppose $z_0 \in \mathbb{H}$ be such that $|z_0| = 1$ and $\frac{-1}{2} < Re(z_0) < 1$. Let γ be the union of the geodesic in $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ joining 0 and z_0 and z_0 and $i\infty$. For $g \in \Gamma_0(pq)\backslash\mathrm{SL}_2(\mathbb{Z})$, let $[g]^*$ be the class of $\pi(g\gamma)$ in $\mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z})$.

We have an intersection pairing

$$\circ : \mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z}) \times \mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z}) \to \mathbb{Z}.$$

Recall the following results of Merel [Prop. 1, Cor. 1, [12]].

Proposition 5. *[14] For* $g, h \in \Gamma_0(pq)\backslash\mathrm{SL}_2(\mathbb{Z})$, *we have*

$$[g]^* \circ [h]_* = 1$$

if $\Gamma_0(pq)g = \Gamma_0(pq)h$ *and*

$$[g]^* \circ [h]_* = 0$$

otherwise.

Corollary 6. *The homomorphism of groups* $\mathbb{Z}^{\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z})} \to \mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z})$ *induced by the map*

$$\xi_0(\sum_g \mu_g g) = \sum_g \mu_g [g]_*$$

is an isomorphism.

The following important property [Cor. 3, [12]] of the intersection pairing will be used later.

Corollary 7. *For* $g \in \Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z})$*, let* $\sum_h \mu_h h \in \mathbb{Z}^{\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z})}$ *be such that* $\sum_h \mu_h [h]_*$ *is the image of an element of* $\mathrm{H}_1(Y_0(pq), \mathbb{Z})$ *under the canonical injection. We have*

$$[g]^* \circ (\sum_h \mu_h [h]_*) = \mu_g.$$

We have a short exact sequence,

$$0 \to \mathrm{H}_1(X_0(pq) - R \cup I, \mathbb{Z}) \to \mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z}) \to \mathbb{Z}^{\{\partial(X_0(pq))\}} \xrightarrow{\delta} \mathbb{Z} \to 0.$$

The boundary map δ takes a geodesic, joining the cusps r and s to the formal symbol $[r] - [s]$. Note that $\delta'(\xi(g)) = \delta([g]^*)$ for all $g \in \mathrm{SL}_2(\mathbb{Z})$.

Recall, we have a canonical bijection $\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (c, d)$. Say α_k, β_r and γ_s are the matrices $\left(\begin{smallmatrix} 0 & -1 \\ 1 & k \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & -r \\ p & rp-1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & -s \\ q & sq-1 \end{smallmatrix}\right)$ respectively. We explicitly write down the elements of $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$ as a set

$$\{(1, k), (1, tp), (1, t'q), (p, q), (q, p), (tp, 1), (t'q, 1), (1, 0), (0, 1)\}$$

with $k \in (\mathbb{Z}/pq\mathbb{Z})^*, t \in (\mathbb{Z}/q\mathbb{Z})^*, t' \in (\mathbb{Z}/p\mathbb{Z})^*$. Observe that $(p, q) = (tp, q) = (p, t'q)$ for all t and t' co prime to pq.

Lemma 8. *The set* $\Omega = \{I, \alpha_k, \beta_r, \gamma_s | 0 \leq k \leq pq - 1, 0 \leq r \leq (p-1), 0 \leq s \leq (q-1)\}$ *forms a complete set of coset representatives of* $\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z})$.

Proof. The orbits $\Gamma_0(pq)\alpha_k$, $\Gamma_0(pq)\beta_l$ and $\Gamma_0(pq)\gamma_m$ are disjoint since ab^{-1} do not belong to $\Gamma_0(pq)$ for two distinct matrices a, b from the set Ω. There are $1+pq+p+q = |\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})|$ coset representatives. □

We list different rational numbers of the form $\alpha(0)$ and $\alpha(\infty)$ with $\alpha \in \Omega$ as equivalence classes of cusps as follows:

0	$\frac{1}{p}$	$\frac{1}{q}$
$\frac{-l}{lp-1}$, $(lp-1, q) = 1$	$\frac{-1}{k}$, $(k, p) > 1$	$\frac{-1}{k}$, $(k, q) > 1$
$\frac{-m}{mq-1}$, $(mq-1, p) = 1$	$\frac{-m}{mq-1}$, $(mq-1, p) > 1$	$\frac{-l}{lp-1}$, $(lp-1, q) > 1$

3.5. **Manin-Drinfeld theorem.** Following [7], we briefly recall the statement of the Manin-Drinfeld theorem.

Theorem 9 (Manin-Drinfeld)**.** *[6] For a congruence subgroup* Γ *and any two cusps* α, β *in* $\mathbb{P}^1(\mathbb{Q})$*, the path*

$$\{\alpha, \beta\} \in \mathrm{H}_1(X_\Gamma, \mathbb{Q}).$$

This theorem can be reformulated in terms of divisor classes on the Riemann surface.

Theorem 10. *Let $a = \sum_i m_i P_i$ be a divisor of degree zero on X. Then a is a divisor of a rational function if and only if there exist a cycle $\sigma \in \mathrm{H}_1(X_\Gamma, \mathbb{Z})$ such that*

$$\int_a \omega = \sum_i m_i \int_{P_0}^{P_i} \omega = \int_\sigma \omega$$

for every $\omega \in \mathrm{H}^0(X_\Gamma, \Omega_{X_\Gamma})$.

As a corollary, we notice that $\{x, y\} \in \mathrm{H}_1(X_\Gamma, \mathbb{Q})$ if and if there is a positive integer m such that $m(\pi_\Gamma(x) - \pi_\Gamma(y))$ is a divisor of a function. In other words, the degree zero divisors supported on the cusps are of finite order in the divisor class group. Manin-Drinfeld proved it using the extended action of the usual Hecke operators. In particular, it says that $\{0, \infty\} \in \mathrm{H}_1(X_\Gamma, \mathbb{Q})$ although 0 and ∞ are two inequivalent cusps of X_Γ. In [17], Ogg constructed certain modular function $X_0(pq)$ whose divisors coincide with degree zero divisors on the modular curves.

4. Eisenstein series for $\Gamma_0(pq)$ with integer coefficients

Let $\sigma_1(n)$ denote the sum of the positive divisors of n. We consider the series

$$E_2'(z) = 1 - 24(\sum_n \sigma_1(n) e^{2\pi i n z}).$$

Let Δ be the Ramanujan's cusp form of weight 12. For all $N \in \mathbb{N}$, the function $z \to \frac{\Delta(Nz)}{\Delta(z)}$ is a function on $\mathbb{H}$ invariant under $\Gamma_0(N)$. The logarithmic differential of this function is $2\pi i E_N(z)dz$ and E_N is a classical holomorphic modular form of weight two for $\Gamma_0(N)$ with constant term $N-1$. The differential form $E_N(z)dz$ is a differential form of third kind on $X_0(N)$. The periods [§ 4.1] of these differential forms are in $\mathbb{Z}$.

By [[5], Thm. 4.6.2], the set $\mathbb{E}_{pq} = \{E_p, E_q, E_{pq}\}$ is a basis of $E_2(\Gamma_0(pq))$.

Lemma 11. *The cusps $\partial(X_0(pq))$ can be identified with the set $\{0, \infty, \frac{1}{p}, \frac{1}{q}\}$.*

Proof. If $\frac{a}{c}$ and $\frac{a'}{c'}$ are in $\mathbb{P}^1(\mathbb{Q})$, then $\Gamma_0(pq)\frac{a}{c} = \Gamma_0(pq)\frac{a'}{c'} \iff \begin{pmatrix} ay \\ c \end{pmatrix} \equiv \begin{pmatrix} a' + jc' \\ c'y \end{pmatrix} \pmod{pq}$, for some j and y such that $\gcd(y, pq) = 1$ [cf. [5], p. 99]. A small check shows that the orbits $\Gamma_0(pq)0$, $\Gamma_0(pq)\infty$, $\Gamma_0(pq)\frac{1}{p}$ and $\Gamma_0(pq)\frac{1}{q}$ are disjoint. $\square$

Let $\mathrm{Div}^0(X_0(pq), \partial(X_0(pq)), \mathbb{Z})$ be the group of degree zero divisors supported on cusps. For all cusps x, let $e_{\Gamma_0(pq)}(x)$ denote the ramification index of x over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ and

$$r_{\Gamma_0(pq)}(x) = e_{\Gamma_0(pq)}(x) a_0(E[x]).$$

By [[18], p. 23], there is a canonical isomorphism $\delta : E_2(\Gamma_0(pq)) \to \mathrm{Div}^0(X_0(pq), \partial(X_0(pq)), \mathbb{Z})$ that takes the Eisenstein series E to the divisor

$$\delta(E) = \sum_{x \in \Gamma_0(pq)\backslash\mathbb{P}^1(\mathbb{Q})} r_{\Gamma_0(pq)}(x)[x]. \tag{4.1}$$

Hence, the Eisenstein element is related to the Eisenstein series by the boundary map. In Prop. 34, we prove that the boundary of Eisenstein element is indeed the boundary of Eisenstein series. By [[19], p. 538], we see that

$$e_{\Gamma_0(pq)}(x) = \begin{cases} q & \text{if } x = \frac{1}{p} \\ p & \text{if } x = \frac{1}{q} \\ 1 & \text{if } x = \infty \\ pq & \text{if } x = 0. \end{cases}$$

Since $\sum_{x \in \partial(X_0(pq))} e_{\Gamma_0(pq)}(x) a_0(E[x]) = 0$, we write the corresponding degree zero divisor as

$$\delta(E) = a_0(E)(\{\infty\} - \{0\}) + q a_0(E[\frac{1}{p}])(\{\frac{1}{p}\} - \{0\}) + p a_0(E[\frac{1}{q}])(\{\frac{1}{q}\} - \{0\}).$$

4.1. **Period Homomorphisms.** We now define the period homomorphisms for the differential forms of third kind.

Definition 12 (Period homomorphism)**.** For $E_N \in \mathbb{E}_{pq}$, the differential forms $E_N(z)dz$ are of third kind on the Riemann surface $X_0(pq)$ but of first kind on the non-compact Riemann surface $Y_0(N)$. For any $z_0 \in \mathbb{H}$ and $\gamma \in \Gamma_0(pq)$, let $c(\gamma)$ be the class in $\mathrm{H}_1(Y_0(pq), \mathbb{Z})$ of the image in $Y_0(pq)$ of the geodesic in $\mathbb{H}$ joining z_0 and $\gamma(z_0)$. That the class is non-zero follows from Thm. 3. This class is independent of the choice of $z_0 \in \mathbb{H}$ and let $\pi_{E_N}(\gamma) = \int_{c(\gamma)} E_N(z)dz$. The map $\pi_{E_N} : \Gamma_0(pq) \to \mathbb{Z}$ is the "period" homomorphism of E_N.

Let $\overline{B}_1(x)$ be the first Bernoulli's polynomial of period 1 defined by

$$\overline{B}_1(0) = 0, \overline{B}_1(x) = x - \frac{1}{2}$$

if $x \in (0, 1)$. For any two integers u and v with $v \geq 1$, we define the Dedekind sum by the formula:

$$S(u, v) = \sum_{t=1}^{v-1} \overline{B}_1(\frac{tu}{v}) \overline{B}_1(\frac{u}{v}).$$

Recall some well-known properties of the period mapping π_{E_N} [cf. [10], p. 10, [14], p. 14] for the Eisenstein series $E_N \in \mathbb{E}_{pq}$.

Proposition 13. *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be an element of* $\Gamma_0(pq)$.

(1) π_{E_N} *is a homomorphism* $\Gamma_0(pq) \to \mathbb{Z}$.

(2) *Consider the number* $\mu = gcd(N-1, 12)$*, the image of* π_{E_N} *lies in* $\mu\mathbb{Z}$.

(3)

$$\pi_{E_N}(\gamma) = \begin{cases} \frac{a+d}{c}(N-1) + 12 sgn(c)(S(d, |c|) - S(d, \frac{|c|}{N})) & \text{if } c \neq 0, \\ \frac{b}{d}(N-1) & \text{if } c = 0. \end{cases}$$

(4)

$$\pi_{E_N}(\gamma) = \pi_{E_N}(\begin{pmatrix} d & \frac{c}{N} \\ Nb & a \end{pmatrix})$$

5. Eisenstein elements

Following [14] and [11], recall the concept of Eisenstein elements of the space of modular symbols. For any natural number $M > 4$, the congruence subgroup $\Gamma_0(M)$ is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $M \mid c$. The congruence subgroup $\Gamma_0(M)$ acts on the upper half plane $\mathbb{H}$ in the usual way. The quotient space $\Gamma_0(M)\backslash\mathbb{H}$ is denoted by $Y_0(M)$. Apriori, these are all Riemann surfaces and hence algebraic curves defined over $\mathbb{C}$. There are models of these algebraic curve defined over $\mathbb{Q}$ and they parametrize elliptic curves with cyclic subgroups of order M. Let $X_0(M)$ be the compactification of the Riemann surface $Y_0(M)$ obtained by adjoining the set of cusps $\partial(X_0(M)) = \Gamma_0(M)\backslash\mathbb{P}^1(\mathbb{Q})$.

Definition 14 (Eisenstein elements)**.** Let $\pi_{E_N} : \mathrm{H}_1(Y_0(pq), \mathbb{Z}) \to \mathbb{Z}$ be the "period" homomorphism of E_N [§ 4.1]. The intersection pairing $\circ$ [11] induces a perfect, bilinear pairing

$$\mathrm{H}_1(X_0(pq), \partial(X_0(pq), \mathbb{Z}) \times \mathrm{H}_1(Y_0(pq), \mathbb{Z}) \to \mathbb{Z}.$$

Since $\circ$ is a non-degenerate bilinear pairing, there is an unique element $\mathcal{E}_{E_N} \in \mathrm{H}_1(X_0(pq), \partial(X_0(pq)), \mathbb{Z})$ such that $\mathcal{E}_{E_N} \circ c = \pi_{E_N}(c)$. The modular symbol $\mathcal{E}_{E_N}$ is the *Eisenstein element* corresponding to the Eisenstein series E_N.

We intersect with the congruence subgroup $\Gamma(2)$ to ensure that the Manin maps become bijective (rather than only surjective), compute the Eisenstein elements for these modular curves, calculate the boundary and show that the these boundaries coincide with the original Eisenstein elements. In the case of $\Gamma_0(p^2)$, although it is difficult to find the Fourier expansion of modular forms at different cusp but fortunately for all $g \in \Gamma_0(p)$ the matrices $g \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) g^{-1}$ belong to $\Gamma_0(p^2)$ and hence it was easier to tackle the explicit coset representatives. Unfortunately, for $N = pq$ or $N = p^3$ these are no longer true.

To get around this problem for the congruence subgroup $\Gamma_0(pq)$ with p and q distinct primes, we use the relative homology group $\mathrm{H}_1(X_0(pq), R \cup I, \mathbb{Z})$. For these relative homology groups, the associated Manin maps are bijective and the push forward of these Eisenstein elements inside the original modular curve turn out to have same boundary as the original Eisenstein elements. We consider three different homology groups in these paper and in particular the study of the relative homology group $\mathrm{H}_1(X_0(N), R \cup I, \mathbb{Z})$ to determine the Eisenstein element is a new idea that we wish to propose in this article. That these relative homology groups should be useful in the study of modular symbol are discovered by Merel.

Definition 15 (Almost eisenstein elements)**.** For $N \in \{p, q, pq\}$, the differential form $E_N(z)dz$ is of first kind on the Riemann surface $Y_0(pq)$. Since $\circ$ is a non-degenerate bilinear pairing, there is an unique element $\mathcal{E}'_{E_N} \in \mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z})$ such that $\mathcal{E}'_{E_N} \circ c = \pi_{E_N}(c)$ for all $c \in \mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z})$. We call $\mathcal{E}'_{E_N}$ the *almost Eisenstein element* corresponding to the Eisenstein series E_N.

6. Even Eisenstein elements

6.1. **Simply connected Riemann surface of genus zero with three marked points.** Recall, there is only one simply connected (genus zero) compact Riemann surface up to conformal bijections: namely the Riemann sphere or the projective complex plane $\mathbb{P}^1(\mathbb{C})$. A theorem of Belyi states that every (compact, connected, non- singular) algebraic curve X has a model defined over $\overline{\mathbb{Q}}$ if and only if it admits a map to $\mathbb{P}^1(\mathbb{C})$ branched over three points.

Consider the subgroup $\Gamma(2)$ of $\mathrm{SL}_2(\mathbb{Z})$ consisting of all matrices which are identity modulo the reduction map modulo 2. The Riemann surface $\Gamma(2) \mod \overline{\mathbb{H}}$ is a Riemann surface of genus zero, denoted by $X(2)$. Hence, it can be identified with $\mathbb{P}^1(\mathbb{C})$.

The subgroup $\Gamma(2)$ has three cusps $\Gamma(2)0$, $\Gamma(2)1$ and $\Gamma(2)\infty$. Hence, $\Gamma(2)\backslash\overline{\mathbb{H}}$ become the simply connected Riemann surface $\mathbb{P}^1(\mathbb{C})$ with the three marked points $\Gamma(2)0$, $\Gamma(2)1$ and $\Gamma(2)\infty$ given by respective cusps. The modular curve $X_0(pq)$ has no obvious morphism to $X(2)$. Hence, we consider the modular curve X_Γ [§6.2]. There are two obvious maps π, π' from X_Γ to the compact Riemann surface $X_0(pq)$.

6.2. **Modular curves with bijective Manin maps.** For the congruence subgroup $\Gamma = \Gamma_0(pq) \cap \Gamma(2)$, consider the compactified modular curve $X_\Gamma = \Gamma\backslash\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ and let $\pi_\Gamma : \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \to X_\Gamma$ be the canonical surjection.

Let $\pi_0 : \Gamma\backslash\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \to \Gamma(2)\backslash\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the map $\pi_0(\Gamma z) = \Gamma(2)z$. The compact Riemann surface $X(2)$ contains three cusps $\Gamma(2)1$, $\Gamma(2)0$, $\Gamma(2)\infty$. Let $P_- = \pi_0^{-1}(\Gamma(2)1)$ and P_+ be the union of two sets $\pi_0^{-1}(\Gamma(2)0)$ and $\pi_0^{-1}(\Gamma(2)\infty)$. Consider now the Riemann surface X_Γ with boundary P_+ and P_-.

Let δ_r be 1 or 0 depending on r is odd or even. For any integer k, let $s_k = (k + (\delta_k - 1)pq)$ be an odd integer. Say l and m be two unique integers such that $lq + mp \equiv 1 \pmod{pq}$ with $1 \leq l \leq (p-1)$ and $1 \leq m \leq (q-1)$. The matrices $\alpha'_{pq} = \left(\begin{smallmatrix} pq & pq-1 \\ pq+1 & pq \end{smallmatrix}\right)$, $\alpha'_k = \left(\begin{smallmatrix} s_k(pq)^2 & s_k pq-1 \\ s_k pq+1 & s_k \end{smallmatrix}\right)$ $\beta'_r = \left(\begin{smallmatrix} -1 & -(r+\delta_r q) \\ p+pq & -1+(r+\delta_r q)(p+pq) \end{smallmatrix}\right)$ and
$\gamma'_s = \left(\begin{smallmatrix} -1 & -(s+\delta_s pq) \\ q+pq & -1+(s+\delta_s pq)(q+pq) \end{smallmatrix}\right)$ are useful to calculate the boundaries of the Eisenstein elements.

Lemma 16. *The set* $\Delta = \{I, \alpha'_k, \beta'_r, \gamma'_s | 0 \leq k \leq (pq-1), 0 \leq r \leq (q-1), 0 \leq s \leq (p-1)\} \subset \Gamma(2)$ *forms an explicit set of coset representatives of* $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$.

Proof. An easy check shows that the orbits $\Gamma_0(pq)\alpha'_k$, $\Gamma_0(pq)\beta'_r$ and $\Gamma_0(pq)\gamma'_s$ are disjoint. Since $|\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})| = pq + p + q + 1$, the result follows. $\square$

The coset representatives in the above lemma are chosen such that $\Gamma_0(pq)\beta_r = \Gamma_0(pq)\beta'_r$ and $\Gamma_0(pq)\gamma_s = \Gamma_0(pq)\gamma'_s$.

Lemma 17. $\Gamma\backslash\Gamma(2)$ *is isomorphic to* $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$

Proof. The explicit closet representatives of Lemma 16 produce the canonical bijection. $\square$

We study the relative homology groups $\mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ and $\mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z})$. The intersection pairing is a non-degenerate bilinear pairing $\circ : \mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z}) \times \mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \to \mathbb{Z}$. Recall the following two fundamental theorems from [14]. For $g \in \Gamma\backslash\Gamma(2)$, let $[g]^0$ (respectively $[g]_0$) be the image in X_Γ of the geodesic in $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ joining $g0$ and $g\infty$ (respectively $g1$ and $g(-1)$). Recall the following two fundamental theorems of [14].

Theorem 18 ([14]). *Let*

$$\xi_0 : \mathbb{Z}^{\Gamma\backslash\Gamma(2)} \to \mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z})$$

be the map which takes $g \in \Gamma\backslash\Gamma(2)$ *to the element* $[g]_0$ *and*

$$\xi^0 : \mathbb{Z}^{\Gamma\backslash\Gamma(2)} \to \mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$$

be the map which takes $g \in \Gamma\backslash\Gamma(2)$ *to the element* $[g]^0$*. The homomorphisms* ξ_0 *and* ξ^0 *are isomorphisms.*

Theorem 19 ([14])**.** *For* $g, g' \in \Gamma(2)$*, we have*

$$[g]_0 \circ [g']^0 = 1$$

if $\Gamma g = \Gamma g'$ *and*

$$[g]_0 \circ [g']^0 = 0$$

otherwise.

The following two lemmas about the set P_- are true for the congruence subgroup $\Gamma_0(N)$ with N odd.

Lemma 20. *We can explicitly write the set* P_- *is of the form* $\Gamma\frac{x}{y}$ *with* x *and* y *both odd.*

Proof. If possible, some element of P_- is of the form $\Gamma\frac{x}{y}$ with x and y co-prime and y even. Consider the corresponding element in the marked simply connected Riemman surface $X(2)$. The cusp $\Gamma(2)\frac{x}{y}$ is an element such that y even and p odd ($\gcd(x,y) = 1$). First, choose p', q' such that $xq' - yp' = 1$ and hence $\left(\begin{smallmatrix} x & p' \\ y & q' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Clearly, q' is odd since y is even. If p' is odd then replace the matrix $\left(\begin{smallmatrix} x & p' \\ y & q' \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} x & p' \\ y & q' \end{smallmatrix}\right) T^{-1}$ to produce a matrix in $\Gamma(2)$ that takes $i\infty$ to $\frac{x}{y}$. This is a contradiction to the fact that $\Gamma\frac{x}{y} \in P_-$.

If x is even then the projection of $\Gamma\frac{x}{y}$ produces an element of $\Gamma(2)0$. Hence, x is necessarily odd. $\square$

The following lemma is deeply influenced by an important propositions of Manin [[8], Prop. 2.2] and Cremona [[4], Prop. 2. 2. 3].

Corollary 21. *We can explicitly write the set* $P_- = \{\Gamma 1, \Gamma\frac{1}{pq}, \Gamma\frac{1}{p}, \Gamma\frac{1}{q}\}$

Proof. Since $P_- = \pi_0^{-1}(\Gamma(2)1)$, we can write every element of the set P_- as $\Gamma\theta 1$ for some $\theta \in \Delta$ (Lemma 16). Let $\delta \in \{1, p, q, pq\}$, then every element of P_- can be written as $\Gamma\frac{u}{v\delta}$ with $\gcd(u, v\delta) = 1$ and $\gcd(v\delta, \frac{pq}{\delta}) = 1$. Choose an odd integer m and an even integer l such that $lu - mv\delta = 1$. A calculation using matrix multiplication shows that $\left(\begin{smallmatrix} 1 & 0 \\ \delta-1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+c & -c \\ c & 1-c \end{smallmatrix}\right) 1 = \frac{1}{\delta}$ and $\left(\begin{smallmatrix} -m & u+m \\ -l & l+v\delta \end{smallmatrix}\right) 1 = \frac{u}{v\delta}$ and hence $A = \left(\begin{smallmatrix} 1 & 0 \\ \delta-1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1+c & -c \\ c & 1-c \end{smallmatrix}\right)\left(\begin{smallmatrix} l+v\delta & -m-u \\ l & -m \end{smallmatrix}\right)$ is a matrix such that $A(\frac{u}{v\delta}) = \frac{1}{\delta}$. The matrix A belongs to Γ if and only if $cv\delta \equiv l' \pmod{\frac{pq}{\delta}}$. Since $v\delta$ is coprime to $\frac{pq}{\delta}$, there is always such c. Hence, the set P_- consists of four elements as in the statement of the Corollary. $\square$

Let $\pi, \pi' : \Gamma\backslash\overline{\mathbb{H}} \to \Gamma_0(pq)\backslash\overline{\mathbb{H}}$ be the maps $\pi(\Gamma z) = \Gamma_0(pq)z$ and $\pi'(\Gamma z) = \Gamma_0(pq)\frac{z+1}{2}$ respectively. Consider the matrix $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. The morphism π' is well-defined since the matrix $h\gamma h^{-1}$ belongs to $\Gamma_0(pq)$ for all $\gamma \in \Gamma$. The morphisms π, π' together induce a map

$$\kappa : \mathbb{C}(X_\Gamma) \to \mathbb{C}(X_0(pq))$$

between the function fields of the Riemann surfaces X_Γ by $\kappa(f(z)) = \frac{f(\pi(\Gamma z))^2}{f(\pi'(\Gamma z))}$. Recall the description of the coordinate chart around a cusp Γx [16] of the Riemann surface X_Γ.

Definition 22. For a cusp y of the congruence subgroup Γ, let Γ_y be the subgroup of Γ fixing the cusp y. Let $t \in \mathrm{SL}_2(\mathbb{R})$ be such that $t(y) = i\infty$ and m be the smallest natural number such that $t\Gamma_y t^{-1}$ is generated by the matrix $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$. For the modular curve X_Γ, the local coordinate around the point Γy is $z \to e^{2\pi i \frac{t(z)}{m}}$.

Example 23. Let $y = \frac{1}{\delta}$ with δ one of the prime p or q, then $h(y) = \frac{u}{\delta}$ with $(u, pq) = 1$. Choose integers u', δ' with δ' even such that $u\delta' - u'\delta = 1$ and hence $\rho_{h(y)} = \left(\begin{smallmatrix} \delta' & u' \\ -\delta & u \end{smallmatrix}\right)$ is a matrix such that $\rho_{h(y)}(h(y)) = i\infty$. We can choose such a $\delta' \in \mathbb{Z}$ since δ is odd.

A calculation using matrix multiplication shows that $\rho_{h(y)} T^e {\rho_{h(y)}}^{-1} = \left(\begin{smallmatrix} 1+e\delta\delta' & e(\delta')^2 \\ -e\delta^2 & 1-e\delta\delta' \end{smallmatrix}\right)$. Hence, the smallest possible e to ensure $tT^e t^{-1} \subset \Gamma_0(pq)$ is $\frac{pq}{\delta}$.

Example 24. Since $\det(\rho_{h(y)} \circ h) = 2$, hence $t = \left(\begin{smallmatrix} \frac{1}{2} & 0 \\ & 1 \end{smallmatrix}\right) \rho_{h(y)} \circ h \in \mathrm{SL}_2(\mathbb{R})$ and $t(y) = i\infty$. A calculation using matrices shows that $tT^e t^{-1} = \begin{pmatrix} 1+\frac{e\delta\delta'}{2} & \frac{e\delta'^2}{4} \\ -e\delta^2 & 1-\frac{e\delta\delta'}{2} \end{pmatrix}$. Hence, the smallest possible e to ensure $tT^e t^{-1} \subset \Gamma$ is $e = 2\frac{pq}{\delta}$.

We use the following lemma to construct differential forms of first kind on the ambient Riemann surface $X_\Gamma - P_+$.

Lemma 25. *Let $f : X_0(pq) \to \mathbb{C}$ be a rational function. The divisors of $\kappa(f)$ are supported on P_+.*

Proof. Suppose f is a meromorphic function on the Riemann surface $X_0(pq)$. Then it is given by $\frac{g}{h}$ with g and h holomorphic function on the Riemann surface $X_0(pq)$. Every element of P_- is of the form $\Gamma\frac{1}{\delta}$ with $\delta \mid N$. By [Prop. 4.1, p. 44, [15]], every holomorphic map on Riemann surface locally looks like $z \to z^n$.

Consider the morphism π' and the point on the modular curve $\Gamma\frac{1}{\delta}$. The local coordinates around the point $\Gamma_0(pq)0, \Gamma_0(pq)\infty$ and $\Gamma_0(pq)\frac{1}{p}$ are given by $q_0(z) = e^{2\pi i \frac{1}{-pqz}}, q_\infty(z) = e^{2\pi i z}$ and $q_{\frac{1}{q}}(z) = e^{2\pi i \frac{z}{p(-qz+1)}}$ respectively. In the modular curve X_Γ, the local coordinates around the points of P_- are given by $q_1(z) = e^{2\pi i \frac{1}{2pq(-z+1)}}$, $q_{\frac{1}{pq}}(z) = e^{2\pi i \frac{z}{2(-pqz+1)}}$, $q_{\frac{1}{p}}(z) = e^{2\pi i \frac{z}{2q(-pz+1)}}$ and $q_{\frac{1}{q}}(z) = e^{2\pi i \frac{z}{2p(-qz+1)}}$.

Now around the point $\Gamma 1$ and $\Gamma\frac{1}{pq}$, we have the equalities $q_0 \circ \pi = q_1^2, q_0 \circ \pi' = q_1^4$ and, $q_{\frac{1}{pq}} \circ \pi = q_{\frac{1}{pq}}^2, q_{\frac{1}{pq}} \circ \pi' = q_{\frac{1}{pq}}^4$.

Let $y = \frac{1}{\delta}$ with δ one of the prime p or q. The local coordinate chart around the point $\Gamma\frac{1}{\delta}$ is $z \to e^{2\pi i \frac{\rho_{h(x)} \circ h(z)}{4e}}$. The map π' takes it to $e^{2\pi i \frac{2\rho_{h(x)}(h(z))}{e}}$. For this coordinate chart the map π' is given by $z \to z^4$.

We now consider the map π and $t = \left(\begin{smallmatrix} 1 & 0 \\ -\delta & 1 \end{smallmatrix}\right)$ is a matrix such that $t(y) = i\infty$ and $e = \frac{pq}{\delta}$. The local coordinate around the point $\Gamma\frac{1}{\delta}$ is $z \to e^{2\pi i \frac{t(z)}{2e}}$ and the map π takes it to $e^{2\pi i \frac{t(z)}{e}}$. In this coordinate chart, the map π is given by $z \to z^2$. Hence, the function $\frac{(f\circ\pi)^2}{f\circ\pi'}$ has no zero or pole on P_-. $\square$

Definition 26. [Even Eisenstein elements] For $E_N \in \mathbb{E}_{pq}$, let $\lambda_{E_N} : X_0(pq) \to \mathbb{C}$ be the rational function whose logarithmic differential is $2\pi i E_N(z)dz = 2\pi i \omega_{E_N}$. Consider the rational function $\lambda_{E_N,2} = \frac{(\lambda_{E_N}\circ\pi)^2}{\lambda_{E_N}\circ\pi'}$ on X_Γ. By Lemma 25, this function has no zeros and poles in P_-. Let $\kappa^*(\omega_{E_N})$ be the logarithmic differential of the function. Let $\varphi_{E_N}(c) = \int_c \kappa^*(\omega_{E_N})$ be the corresponding "period" homomorphism $\mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z}) \to \mathbb{Z}$.

By the non-degeneracy of the intersection pairing, there is a unique element $\mathcal{E}^0_{E_N} \in \mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ such that $\mathcal{E}^0_{E_N} \circ c = \varphi_{E_N}(c)$ for all $c \in \mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z})$. The modular symbol $\mathcal{E}^0_{E_N}$ is the *even Eisenstein element* corresponding to the Eisenstein series E_N.

For $E_N \in \mathbb{E}_{pq}$, define a function $F_{E_N} : \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z}) \to \mathbb{Z}$ by

$$F_{E_N}(g) = \varphi_{E_N}(\xi_0(g)) = \int_{g(1)}^{g(-1)} [2E_N(z) - E_N(\frac{z+1}{2})]dz.$$

Remark 27. It is easy to see that for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2), h\gamma h^{-1} = \left(\begin{smallmatrix} a+c & \frac{b+d-a-c}{2} \\ 2c & d-c \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

For any matrix $\gamma \in \Gamma$, consider the rational number $P_N(\gamma) = \frac{2\pi_{E_N}(\gamma) - \pi_{E_N}(h\gamma h^{-1})}{12}, t(\gamma) = b+d-a-c$ and $s(\gamma) = a+c$.

Lemma 28. *For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ with $c \neq 0$,*

$$\begin{aligned} P_N(\gamma) = sgn(t(\gamma))[2(S(s(\gamma), |t(\gamma)|pq) - S(s(\gamma), |t(\gamma)|)) \\ -S(s(\gamma), |\frac{t(\gamma)}{2}|pq) + S(s(\gamma), \frac{|t(\gamma)|}{2})]. \end{aligned}$$

In particular, $P_N(\gamma) \in \mathbb{Z}$ for all $\gamma \in \Gamma$.

Proof. Recall the properties of period homomorphism [cf. Prop. 13]. We calculate the corresponding periods

$$\pi_{E_N}(\gamma) = \pi_E(T\gamma T^{-1}) = \pi_{E_N}(\begin{pmatrix} a+c & -(a+c)+b+d \\ c & -c+d \end{pmatrix}) = \pi_{E_N}(\begin{pmatrix} a+c & -(a+c)+b+d \\ c & -c+d \end{pmatrix}) = \pi_{E_N}(\begin{pmatrix} d-c & \frac{c}{N} \\ t(\gamma)N & a+c \end{pmatrix}).$$

By [cf. Prop. 13], we have

$$\pi_{E_N}(\gamma) = \frac{a+d}{t(\gamma)N}(N-1) + 12sgn(t(\gamma))[(S(s(\gamma), |t(\gamma)|N) - S(s(\gamma), |t(\gamma)|]$$

Similarly,

$$\begin{aligned} \pi_{E_N}(h\gamma h^{-1}) = \pi_{E_N}(\begin{pmatrix} a+c & \frac{b+d-a-c}{2} \\ 2c & d-c \end{pmatrix}) = \pi_{E_N}(\begin{pmatrix} d-c & \frac{2c}{N} \\ \frac{t(\gamma)N}{2}N & a+c \end{pmatrix}) \\ = \frac{2(a+d)}{t(\gamma)N}(N-1) + 12sgn(t(\gamma))[S(s(\gamma), \frac{|t(\gamma)|}{2}N) - S(s(\gamma), \frac{|t(\gamma)|}{2}))] \end{aligned}$$

Hence, we deduce the formula as in the statement. From the formula, it is easy to see that $P_N(\gamma) \in \mathbb{Z}$ for all $\gamma \in \Gamma$. □

Let x be one of the prime p or q. Choose integers s, s' and l, l' such that $l(s_kx+2) - 2spq = 1$ and $l's_kx - 2s'\frac{pq}{x} = 1$. Let $\gamma_1^{x,k} = \begin{pmatrix} 1+4spq & -2l \\ -4s(s_kx+2)pq & 1+4spq \end{pmatrix}$ and $\gamma_2^{x,k} = \begin{pmatrix} 1+4s'\frac{pq}{x} & -2l' \\ -4s'(s_k)pq & 1+4s'\frac{pq}{x} \end{pmatrix}$ be two matrices in Γ. Since the integers l and l' are necessarily odd, we have $\gamma_1^{x,k}(\frac{1}{s_kx+2}) = -\frac{1}{s_kx+2}$ and $\gamma_2^{x,k}(\frac{1}{s_kx}) = -\frac{1}{s_kx}$.

Using the formula of Lemma 28, we deduce that

$$s(\gamma_1^{x,k}) = 1 - 4spq(1+s_kx), t(\gamma_1^{x,k}) = -2(l - 2s(s_kx+2)pq)$$

and

$$s(\gamma_2^{x,k}) = 1 - 4s'pq(s_k - \frac{1}{x}), t(\gamma_2^{x,k}) = -2(l' - 2s's_kpq).$$

We can now calculate $P_N(\gamma_1^{x,k}), P_N(\gamma_2^{x,k})$ using Lemma 28.

Proposition 29.

$$F_{E_N}(g)=\begin{cases}12(S(r,N)-2S(r,2N)) & \textit{if } g=(r-1,r+1),\\ 6(P_N(\gamma_1^{x,k})-P_N(\gamma_2^{x,k})) & \textit{if } g=(1+kx,1) \textit{ or } g=(-1-kx,1),\\ -6(P_N(\gamma_1^{x,k})-P_N(\gamma_2^{x,k})) & \textit{if } g=(1,-1-kx) \textit{ or } g=(1,1+kx),\\ 0 & \textit{if } g=(\pm1,1).\end{cases}$$

Proof. If $g=(r-1,r+1)$ and $E_N\in\mathbb{E}_{pq}$, we get [[14], p. 18]

$$F_{E_N}(g)=\varphi_{E_N}(\xi_0(g))=12(S(r,N)-2S(r,2N)).$$

We proceed to find the value of the integrals in the remaining cases. The differential form $k^*(\omega_{E_N})$ is of first kind on the Riemann surface $X_\Gamma-P_+$. We also note that $g=(\pm1,1),(\pm1\pm kx,1)$ or $(1,\pm1\pm kx)$ with x one of the prime p or q, then we can't write it as $(r-1,r+1)$.

Since all the Fourier coefficients of the Eisenstein series are real valued, so an argument similar to [[14], p. 19] shows that $F_{E_N}(s_kx+1,1)=F_{E_N}(-s_kx-1,1)$. Consider the path $\{\frac{1}{s_kx+2},-\frac{1}{s_kx+2}\}=\{\frac{1}{s_kx+2},\frac{1}{s_kx}\}+\{\frac{1}{s_kx},\frac{-1}{s_kx}\}+\{\frac{-1}{s_kx},\frac{-1}{s_kx+2}\}$. The rational number $\frac{1}{s_kx}$ correspond to a point of P_- in the Riemann surface X_Γ. The differential form $k^*\omega_{E_N}$ has no zeros and poles on P_-. We deduce that

$$\int_{\frac{1}{s_kx+2}}^{-\frac{1}{s_kx+2}}k^*(\omega_{E_N})=\int_{\frac{1}{s_kx+2}}^{\frac{1}{s_kx}}k^*(\omega_{E_N})+\int_{\frac{1}{s_kx}}^{\frac{-1}{s_kx}}k^*(\omega_{E_N})+\int_{\frac{-1}{s_kx}}^{\frac{-1}{s_kx+2}}k^*(\omega_{E_N})=2F_N(s_kx+1,1)+\int_{\frac{1}{s_kx}}^{\frac{-1}{s_kx}}k^*(\omega_{E_N}).$$

Let $\gamma_1^{x,k}$ and $\gamma_2^{x,k}$ be two matrices in Γ such that $\gamma_1^{x,k}(\frac{1}{s_kx+2})=-\frac{1}{s_kx+2}$ and $\gamma_2^{x,k}(\frac{1}{s_kx})=-\frac{1}{s_kx}$. We deduce that $2F_N(s_kx+1,1)=\int_{\frac{1}{s_kx+2}}^{\gamma_1^{x,k}(\frac{1}{s_kx+2})}k^*(\omega_{E_N})-\int_{\frac{1}{s_kx}}^{\gamma_2^{x,k}(\frac{1}{s_kx})}k^*(\omega_{E_N})$.

We now prove that the $\int_{\frac{1}{s_kx}}^{\gamma_2^{x,k}(\frac{1}{s_kx})}k^*(\omega_{E_N})$ is independent of the choice of the matrices $\gamma_2^{x,k}\in\Gamma$ that take $\frac{1}{s_kx}$ to $-\frac{1}{s_kx}$. If possible, $\gamma_2^{x,k}$ and ${\gamma'}_2^{x,k}$ be two matrices such that $\gamma_2^{x,k}(\frac{1}{s_kx})={\gamma'}_2^{x,k}(\frac{1}{s_kx})=-\frac{1}{s_kx}$. Since $\gamma_2^{x,k}\in\Gamma$, the integral $\varphi_{E_N}(\gamma_2^{x,k})=\int_{\frac{1}{s_kx}}^{\gamma_2^{x,k}(\frac{1}{s_kx})}k^*(\omega_{E_N})$ is independent of the choice of any point in $\mathbb{H}\cup\{-1\}$, hence by replacing $\frac{1}{s_kx}$ with $(\gamma_2^{x,k})^{-1}({\gamma'}_2^{x,k})\frac{1}{s_kx}$, we get the above integral is same as $\int_{\frac{1}{s_kx}}^{{\gamma'}_2^{x,k}(\frac{1}{s_kx})}k^*(\omega_{E_N})$ and the integral is independent of the choice of exceptional matrices. Similarly, we can prove that $\int_{\frac{1}{s_kx+2}}^{\gamma^{x,k}(\frac{1}{s_kx+2})}k^*(\omega_{E_N})$ is also independent of the choice of the matrices that take $\frac{1}{s_kx+2}$ to $-\frac{1}{s_kx+2}$. Since we have already written down two matrices $\gamma_1^{x,k}$ and $\gamma_2^{x,k}$ in Γ such that $\gamma_1^{x,k}(\frac{1}{s_kx+2})=-\frac{1}{s_kx+2}$ and $\gamma_2^{x,k}(\frac{1}{s_kx})=-\frac{1}{s_kx}$, we use these matrices to find those integrals.

The above calculation shows that

$$2\pi_{E_N}(\gamma_1^{x,k})-\pi_{E_N}(h\gamma_1^{x,k}h^{-1})=2F_N(s_kx+1,1)+2\pi_{E_N}(\gamma_2^{x,k})-\pi_{E_N}(h\gamma_2^{x,k}h^{-1}).$$

Hence, we get

$$F_{E_N}(s_kx+1,1)=\frac{2\pi_{E_N}(\gamma_1^{x,k})-\pi_{E_N}(h\gamma_1^{x,k}h^{-1})-2\pi_E(\gamma_2^{x,k})+\pi_E(h\gamma_2^{x,k}h^{-1})}{2}=6(P_N(\gamma^{x,k})-P_N(\gamma_2^{x,k})).$$

Since $F_{E_N}((1+s_kx,1))=-F_{E_N}((1,-1-s_kx))$, the above equation determine the Eisenstein elements for the Eisenstein series E_N completely. $\square$

From the above lemma, we conclude that $6F_N(g) = F_{E_N}(g)$.

Lemma 30. *For $E_N \in E_2(\Gamma_0(pq))$, let us consider the element $\mathcal{E}^0_{E_N}$ of $\mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ defined by $\mathcal{E}^0_{E_N} = \sum_{g\in\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} F_{E_N}(g)\xi^0(g)$. For all $c \in \mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z})$, we have $\mathcal{E}^0_{E_N} \circ c = \varphi_{E_N}(c)$*

Proof. By Theorem 19, we write the even Eisenstein element uniquely as

$$\sum_{g\in\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} H_{E_N}(g)\xi^0(g).$$

By loc. cit. , $[g]_0 \circ [h]^0 = 1$ if and only if $\Gamma g = \Gamma h$. The function H_{E_N} and F_{E_N} coincide since $H_{E_N}(g) = \sum_{g\in\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} H_{E_N}(g)\xi^0(g) \circ \xi_0(g) = \mathcal{E}^0_{E_N} \circ \xi_0(g) = F_{E_N}(g)$. □

For the modular curve X_Γ, we have a similar short exact sequence

$$0 \to \mathrm{H}_1(X_\Gamma - P_-, \mathbb{Z}) \to \mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \xrightarrow{\delta^0} \mathbb{Z}^{P_+} \to \mathbb{Z} \to 0.$$

The boundary map δ^0 takes a geodesic, joining the the point r and s of P_+ to the formal symbol $[r]-[s]$.

7. Eisenstein elements and winding elements for $\Gamma_0(pq)$

7.1. **Eisenstein elements for $\Gamma_0(pq)$.** We first prove an elementary number theoretic lemma. Recall, l and m are two unique integers such that $lq + mp \equiv 1 \pmod{pq}$ with $1 \leq l \leq (p-1)$ and $1 \leq m \leq (q-1)$.

Lemma 31. *For all k with $1 \leq k \leq (q-1)$, we can choose an integer $s(k) \in (\mathbb{Z}/q\mathbb{Z})$ such that*

$$(kp, -1) = (p, s(k)p - 1)$$

in $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$. The map $k \to s(k)$ is a bijection $(\mathbb{Z}/q\mathbb{Z})^* \to (\mathbb{Z}/q\mathbb{Z}) - \{\overline{m}\}$.

Proof. For all k with $1 \leq k \leq (q-1)$, let k' be the inverse of k in $(\mathbb{Z}/q\mathbb{Z})^*$. By Chinese remainder theorem, we choose an unique x with $1 \leq x \leq (pq-1)$ such that $x \equiv -1 \pmod{p}$ and $x \equiv -k' \pmod{q}$. Observe that x is coprime to both p and q. We write $x = s(k)p-1$ for an unique $s(k)$ with $0 \leq s(k) \leq (q-1)$. Since $\Gamma_0(pq)\backslash\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$, we deduce that $(kp, -1) = (xkp, -x) = (-p, -x) = (p, x) = (p, s(k)p - 1)$ in $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$.

Consider the map $(\mathbb{Z}/q\mathbb{Z})^* \to (\mathbb{Z}/q\mathbb{Z})$ given by $k \to s(k)$. If $lq + mp \equiv 1 \pmod{pq}$ then m is not in the image of this map. This map is one-one since if $s(k) = s(h)$ then $k \equiv h \pmod{q}$. Hence, the map $(\mathbb{Z}/q\mathbb{Z})^* \to (\mathbb{Z}/q\mathbb{Z}) - \{\overline{m}\}$ $k \to s(k)$ is a bijection. □

For all t coprime to pq, consider the set V of all matrices of the form α_t.

Proposition 32. *The boundary of any element*

$$X = \sum_{g\in\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} F(g)[g]^*$$

in $\mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z})$ is of the form

$$\delta(X) = A(X)[\frac{1}{p}] + B(X)[\frac{1}{q}] + C(X)[\infty] - (A(X) + B(X) + C(X)[0]$$

with

$$A(X)=\sum_{k=0}^{q-1}[F(\beta_k)-F(\beta_k S)], B(X)=\sum_{i=0}^{p-1}[F(\gamma_i)-F(\gamma_i S)]$$

and $C(X)=[F(0,1)-F(1,0)]$.

Proof. Choose an explicit coset representatives of $\Gamma_0(pq)\backslash \mathrm{SL}_2(\mathbb{Z})$ (cf. Lemma 8) and write

$$X=C(X)[I]^*+\sum_{\alpha_t\in V}F(1,t)[\alpha_t]^*+\sum_{k=1}^{q-1}F(1,kp)[\alpha_{kp}]^*+\sum_{k=1}^{p-1}F(1,kq)[\alpha_{kq}]^*$$

$$+\sum_{i=0}^{q-1}F(p,ip-1)[\beta_i]^*+\sum_{j=0}^{p-1}F(q,jq-1)[\beta_j]^*.$$

According to Lemma 31 for $1\leq k\leq (q-1)$, we have $\alpha_{kp}S=Z\beta_{s(k)}$ for some $Z\in\Gamma_0(pq)$. We deduce that

$$\sum_{k=1}^{q-1}F(1,kp)[\alpha_{kp}]^*+\sum_{i=0}^{q-1}F(p,ip-1)[\beta_i]^*=\sum_{k=1}^{q-1}(F(1,kp)[\alpha_{kp}]^*+F(kp,-1)[\alpha_{kp}S]^*)+F(\beta_m)[\beta_m]^*$$

and

$$\sum_{k=1}^{p-1}F(1,kq)[\alpha_{kq}]^*+\sum_{j=0}^{p-1}F(q,jq-1)[\gamma_j]^*=\sum_{k=1}^{p-1}(F(1,kq)[\alpha_{kq}]^*+F(kq,-1)[\alpha_{kq}S]^*)+F(\gamma_l)[\gamma_l]^*.$$

A small check shows that $\delta([\alpha_{kp}]^*)=\delta([\alpha_p]^*)$ and $\delta([\alpha_{kp}]^*)=-\delta([\alpha_{kp}S]^*)$.

We now calculate $\delta([\beta_m]^*)$ and $\delta([\gamma_l]^*)$. Since $lq+mp\equiv 1 \pmod{pq}$ and $-I\in\Gamma_0(pq)$, so we get

$$\begin{pmatrix} 1-q(l-1) & m(l-1)\\ (l-1)pq & 1+lq(l-1)\end{pmatrix}\begin{pmatrix} m & -l\\ q & p\end{pmatrix}=\gamma\beta_m S \tag{7.1}$$

and

$$\begin{pmatrix} 1-p(m+1) & -l(m+1)\\ (1+m)pq & 1-mp(l+m)\end{pmatrix}\begin{pmatrix} m & -l\\ q & p\end{pmatrix}=\begin{pmatrix} -1 & -l\\ q & -mp\end{pmatrix}=\gamma_l,$$

for some $\gamma\in\Gamma_0(pq)$ and hence we have $\Gamma_0(pq)\beta_m S=\gamma_l$. From $\delta([\beta_m]^*)=\delta([\alpha_q]^*-[\alpha_p]^*)$ and $\delta([\gamma_l]^*)=\delta([\alpha_p]^*-[\alpha_q]^*)$, it is easy to see that

$$\delta(\sum_{k=1}^{q-1}F(1,kp)[\alpha_{kp}]^*+\sum_{i=0}^{q-1}F(p,jp-1)][\beta_j]^*)=\sum_{k=1}^{q-1}[F(1,kp)-F(kp,-1)]\delta([\alpha_p]^*)+F(\beta_m)\delta([\beta_m]^*).$$

and

$$\delta(\sum_{k=1}^{p-1}F(1,kq)[\alpha_{kq}]^*+\sum_{j=0}^{p-1}F(q,jq-1)][\gamma_j]^*)=\sum_{k=1}^{p-1}[F(1,kq)-F(kq,-1)]\delta([\alpha_q]^*)+F(q,lq-1)\delta([\gamma_l]^*).$$

$$F(p,mp-1)\delta([\beta_m]^*)+F(q,lq-1)\delta([\gamma_l]^*)=[F(\beta_m)-F(\beta_m S)](\delta([\alpha_q]^*)-\delta([\alpha_p]^*)).$$

Recall, $\delta([\alpha_p]^*)=[0]-[\frac{1}{p}]$ and $\delta([\alpha_q]^*)=[0]-[\frac{1}{q}]$. The above calculation shows that $\delta(X)=C(X)\delta([I]^*)+A(X)\delta([\alpha_p]^*)+B(X)\delta([\alpha_q]^*)$ with, $A(X)=\sum_{k=0}^{q-1}[F(p,kp-1)-F(kp-1,-p)]$, $B(X)=\sum_{m=0}^{p-1}[F(\gamma_l')-F(\gamma_l'S)]$ and $C(X)=F(I)-F(S)$. We deduce the proposition. $\square$

We also prove similar proposition for $\Gamma\subset\Gamma(2)$.

Proposition 33. *The boundary of any element*

$$X = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} F(g)\xi^0(g)$$

in $\mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ *is of the form*

$$\delta^0(X) = A'(X)[\frac{1}{p}] + B'(X)[\frac{1}{q}] + C'(X)[\infty] - (A'(X) + B'(X) + C'(X)[0]$$

with

$$A'(X) = \sum_{k=0}^{q-1}[F(\beta'_k) - [\sum_{k=1}^{q-1} F(\alpha'_{kp})] - F(\gamma'_l), B'(X) = \sum_{i=0}^{p-1}[F(\gamma'_i) - [\sum_{k=1}^{p-1} F(\alpha'_{kq})] - F(\beta'_m)$$

and $C'(X) = [F(0,1) - F(\alpha'_{pq})]$.

Proof. This is a straightforward calculation using the coset representatives of $\Gamma\backslash\Gamma(2)$ [cf. Lemma 16]. $\square$

Proposition 34. *For* $E \in \mathbb{E}_{pq}$*, the boundaries of almost Eisenstein elements* $\mathcal{E}'_E$ *in* $\mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z})$ *corresponding to the Eisenstein series* E *are* $-\delta(E)$ *[§4].*

Proof. For $E \in \mathbb{E}_{pq}$, let $\mathcal{E}'_E = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} G_E(g)[g]^*$ be the almost Eisenstein element. According to Proposition 32, we need to calculate $A(\mathcal{E}'_E)$, $B(\mathcal{E}'_E)$ and $C(\mathcal{E}'_E)$.

For all $0 \leq k < (q-1)$, $\beta_k T = \beta_{k+1}$ and $\beta_{q-1}T = \gamma\beta_0$ with $\gamma = \begin{pmatrix} 1+pq & q \\ -qp^2 & 1-qp \end{pmatrix}$. We have an inclusion $\mathrm{H}_1(Y_0(pq), \mathbb{Z}) \to \mathrm{H}_1(Y_0(pq), R \cup I, \mathbb{Z})$. Since $\{\rho^*, \gamma\rho^*\} = \{\beta_0\rho^*, \gamma\beta_0\rho^*\} = -\sum_{k=0}^{q-1}\{\beta_k\rho, \beta_k\rho^*\}$, we deduce that

$$\pi_E(\gamma) = \int_{z_0}^{\gamma z_0} E(z)dz = \mathcal{E}'_E \circ \{z_0, \gamma z_0\} = -\mathcal{E}'_E \circ (\sum_{k=0}^{q-1}\{\beta_k\rho, \beta_k\rho^*\}) = -\sum_{k=0}^{q-1} \mathcal{E}'_E \circ \{\beta_k\rho, \beta_k\rho^*\}.$$

Applying Cor. 6, we have $\sum_{k=0}^{q-1} \mathcal{E}'_E \circ \{\beta_k\rho, \beta_k\rho^*\} = \sum_{k=0}^{q-1}[G_E(\beta_k) - G_E(\beta_k S)] = -A(\mathcal{E}'_E)$. Hence, we prove that $A(\mathcal{E}'_E) = -\pi_E(\gamma)$. By interchanging p and q, we have $B(\mathcal{E}'_E) = -\pi_E(\gamma_0)$ for $\gamma_0 = \begin{pmatrix} 1+pq & p \\ -pq^2 & 1-qp \end{pmatrix}$.

We now calculate $\pi_E(\gamma)$ and $\pi_E(\gamma_0)$ using [19]. Recall, $\frac{1}{p}$ is a cusp with $e_{\Gamma_0(pq)}(\frac{1}{p}) = q$. Consider the matrices $x = \begin{pmatrix} 1 & -q \\ -p & 1+qp \end{pmatrix}$ and $y = \begin{pmatrix} 1 & -p \\ -q & 1+qp \end{pmatrix}$ respectively. One can easily check that $x \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} x^{-1} = \gamma$ and $y \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} y^{-1} = \gamma_0$. Notice that $x(i\infty) = \Gamma_0(pq)\frac{1}{p}$ and $y(i\infty) = \Gamma_0(pq)\frac{1}{q}$. By [[19], p. 524], we deduce that $\pi_E(\gamma) = e_{\Gamma_0(pq)}(\frac{1}{q})a_0(E[\frac{1}{p}])$ and $\pi_{E_{pq}}(\gamma_0) = e_{\Gamma_0(pq)}(\frac{1}{p})a_0(E[\frac{1}{p}])$.

According to Proposition 32, the boundary of the almost Eisenstein element corresponding to an Eisenstein series E is

$$\delta(\mathcal{E}'_E) = A(\mathcal{E}'_E)[\frac{1}{p}] + B(\mathcal{E}'_E)[\frac{1}{q}] + C(\mathcal{E}'_E)[\infty] - (A(\mathcal{E}'_E) + B(\mathcal{E}'_E) + C(\mathcal{E}'_E))[0]$$

with $A(\mathcal{E}'_E) = qa_0(E[\frac{1}{p}])$, $B(\mathcal{E}'_E) = pa_0(E[\frac{1}{q}])$ and $C(\mathcal{E}'_E) = -[F(I) - F(S)]$. Applying Cor. 6 again, we deduce that that $F(I) - F(S) = \int_\rho^{\rho^*} E(z)dz = -a_0(E)$. For $E \in E_2(\Gamma_0(pq))$, the boundary of E is

$$\delta(E) = a_0(E)([\infty] - [0]) + qa_0(E[\frac{1}{p}])([\frac{1}{p}] - [0]) + pa_0(E[\frac{1}{q}])([\frac{1}{q}] - [0]) = \delta(\mathcal{E}'_E).$$

$\square$

Let β and h be the matrices $\left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&2\end{smallmatrix}\right)$ respectively. Let

$$\pi_* : \mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \to \mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq)), \mathbb{Z})$$

be the isomorphism defined by $\pi_*(\xi_0(g)) = [g]^*$ [[12], Cor. 1]. It is easy to see that $\delta(\pi_*(X)) = \delta^0(X)$ for all $X \in \mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$

Proposition 35. *For all $E \in \mathbb{E}_{pq}$, let $\mathcal{E}_E^0$ be the even Eisenstein element in $\mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ [§ 6]. The boundary of the modular symbol $\pi_*(\mathcal{E}_E^0)$ is $-6\delta(E)$.*

Proof. By Theorem 12, we explicitly write down the even Eisenstein element $\mathcal{E}_E^0$ in the relative homology group $\mathrm{H}_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ as

$$\mathcal{E}_E^0 = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})} F_E(g)\xi_0(g).$$

According to Proposition 19, we need to calculate $A'(\mathcal{E}_E^0), B'(\mathcal{E}_E^0)$ and $C'(\mathcal{E}_E^0)$. For $0 \le k < (q-2)$, we have $\beta'_k\beta = \beta'_{k+2}$. A small check shows that $\beta'_{q-1}\beta = \beta'_1$ and $\beta'_{q-2}\beta = \gamma'\beta'_0$ with

$$\gamma' = \left(\begin{smallmatrix} 1+2pq(1+q) & 2q \\ -2q(p+pq)^2 & 1-2pq(1+q) \end{smallmatrix}\right) \in \Gamma.$$

As a homology class in $\mathrm{H}_1(X_\Gamma - P_+, P_-, \mathbb{Z})$, we have

$$\{-1, \gamma'(-1)\} = \{\beta'_0(-1), \gamma'\beta'_0(-1)\} = -\sum_{k=0}^{q-1}\{\beta'_k(1), \beta'_k(-1)\} = \sum_{k=0}^{q-1}\{\beta'_k(-1), \beta'_k(1)\}.$$

By the definition of the even Eisenstein elements, we conclude that

$$\int_{z_0}^{\gamma' z_0} k^*(\omega_E) = \mathcal{E}_E^0 \circ \{z_0, \gamma' z_0\} = -\mathcal{E}_E^0 \circ (\sum_{k=0}^{q-1}(\beta'_k(1), \beta'_k(-1)) = -\sum_{k=0}^{q-1} \mathcal{E}_E^0 \circ \{\beta'_k(1), \beta'_k(-1)\}.$$

It is easy to see that $hASBh^{-1} \in \mathrm{SL}_2(\mathbb{Z})$ for all $A, B \in \Gamma(2)$. Since $[\alpha'_{kq}S] = [\gamma'_{s(k)}]$ in $\mathbb{P}^1(\mathbb{Z}/pq\mathbb{Z})$, so $\kappa' = \alpha'_{kq}S(\gamma'_{s(k)})^{-1} \in \Gamma_0(pq)$ and $h\kappa' h^{-1} \in \Gamma_0(pq)$. We deduce that the differential form

$$k^*(\omega_E) = f(z)dz = [2E(z) - \frac{1}{2}E(\frac{z+1}{2})]dz$$

is invariant under κ'. According to the above argument,

$$F_E(\alpha'_{kq}) = \int_{\alpha'_{kq}(1)}^{\alpha'_{kq}(-1)} f(z)dz = \int_{\alpha'_{kq}S(-1)}^{\alpha'_{kq}S(1)} f(z)dz = -\int_{\alpha'_{kq}S(1)}^{\alpha'_{kq}S(-1)} f(z)dz \tag{7.2}$$

$$= -\int_{\kappa'^{-1}\alpha'_{kq}S(1)}^{\kappa'^{-1}\alpha'_{kq}S(-1)} f(\kappa' z)d\kappa' z = -\int_{\gamma'_{s(k)}(1)}^{\gamma'_{s(k)}(-1)} f(z)dz = -F_E(\gamma'_{s(k)}).$$

A similar calculation shows that $F_E(\gamma'_l) = -F_E(\beta'_m)$ and $F_E(\alpha_{kp}) = -F_E(\beta_{s(k)})$ for some $s(k) \in (\mathbb{Z}/q\mathbb{Z})^*$. Applying Theorem 18, we have

$$\sum_{k=0}^{q-1} F_E(\beta'_k) = \sum_{k=0}^{q-1} \mathcal{E}_E^0 \circ \{\beta'_k(1), \beta'_k(-1)\} = -\int_{z_0}^{\gamma' z_0} k^*(\omega_E).$$

According to the definition of the period π_E of the Eisenstein series $E(z)$ [cf. Section 3], we get

$$\int_{z_0}^{\gamma' z_0} k^*(\omega_E) = \int_{z_0}^{\gamma' z_0} [2E(z) - \frac{1}{2}E(\frac{z+1}{2})]dz = 2\pi_E(\gamma') - \pi_E(h\gamma' h^{-1}).$$

We calculate $\pi_E(\gamma')$ and $\pi_E(h\gamma' h^{-1})$. From 27, it is easy to see that $h\gamma' h^{-1} = \left(\begin{smallmatrix} 1+z & qv^2 \\ -4p^2q(1+q)^2 & 1-z \end{smallmatrix}\right)$ with $v = (1-p(1+q))$ and $z = 2pqv(1+q)$. Furthermore, the matrix $h\gamma' h^{-1}$ decomposes as

$$h\gamma' h^{-1} = \left(\begin{smallmatrix} 1-p(1+q) & \frac{p(1+q)}{2} \\ -2p(1+q) & 1+p(1+q) \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1-p(1+q) & \frac{p(1+q)}{2} \\ -2p(1+q) & 1+p(1+q) \end{smallmatrix}\right)^{-1}.$$

Since the matrix $\left(\begin{smallmatrix} 1-p(1+q) & \frac{p(1+q)}{2} \\ -2p(1+q) & 1+p(1+q) \end{smallmatrix}\right)^{-1}$ takes the cusp $i\infty$ to $\frac{1}{p}$, we have $\pi_E(h\gamma' h^{-1}) = qa_0(E[\frac{1}{p}])$. We further decompose γ' as

$$\left(\begin{smallmatrix} 1 & -2q \\ -p(1+q) & 1+2pq(1+q) \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 2q \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & -2q \\ -p(1+q) & 1+2pq(1+q) \end{smallmatrix}\right)^{-1}.$$

The matrix $\left(\begin{smallmatrix} 1 & -2q \\ p(1+q) & 1+2pq(1+q) \end{smallmatrix}\right)$ takes the cusp $i\infty$ to $\frac{1}{p}$. We deduce that $\pi_E(\gamma') = 2qa_0(E[\frac{1}{p}])$ and $\int_{z_0}^{\gamma' z_0} k^*(\omega_E) = 3a_0(E[\frac{1}{p}])$. A simple calculation shows that

$$A'(\mathcal{E}_E^0) = \sum_{k=0}^{q-1} F_E(\beta'_k) - \sum_{k=0}^{q-1} F_E(\alpha'_{kp}) - F_E(\gamma'_m) = 2\sum_{k=0}^{q-1} F_E(\beta'_k) = -6a_0(E[\frac{1}{p}]).$$

By interchanging p and q, we obtain $B'(\mathcal{E}_E^0) = -6a_0(E[\frac{1}{q}])$. Since $\alpha'_{pq}S \in \Gamma_0(pq)$, a calculation similar to Equation 7.2 shows that

$$F_E(I) = -F_E(\alpha_{pq}) = \int_1^{-1} [2E(z) - \frac{1}{2}E(\frac{z+1}{2})]dz = -\int_{-1}^{\beta(-1)} [2E(z) - \frac{1}{2}E(\frac{z+1}{2})]dz = -3a_0(E),$$

we conclude that $C'(\mathcal{E}_E^0) = [F_E(I) - F_E(\alpha_{pq})] = -6a_0(E)$ and hence $\delta^0(\mathcal{E}_E^0) = \delta(\mathcal{E}_E^0) = -6\delta(E)$. $\square$

The inclusion map $i : (X_0(pq) - R \cup I, \partial(X_0(pq)) \to (X_0(pq), \partial(X_0(pq))$ induces an onto map $i_* : \mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq), \mathbb{Z}) \to \mathrm{H}_1(X_0(pq), \partial(X_0(pq)), \mathbb{Z})$ with $i_*([g]^*) = \xi(g)$. Note that $\delta([g]^*) = [g.0] - [g.\infty] = \delta'(\xi(g)) = \delta'(i_*([g]^*))$. From [§ 3.4], we have $\delta(c) = \delta'(i_*(c))$ for all homology class $c \in \mathrm{H}_1(X_0(pq) - R \cup I, \partial(X_0(pq), \mathbb{Z})$.

Lemma 36. *The integrals of every holomorphic differential on $X_0(pq)$ over $i_*(\mathcal{E}'_E)$ and $i_*\pi_*(\mathcal{E}_E^0)$ are zero.*

Proof. A straightforward generalization of [[14], Lemma 5]. $\square$

We now prove the main Theorem 1 of this article.

Proof. By [[12], Cor. 3], we obtain $i_*(\mathcal{E}'_E) \circ c = \mathcal{E}'_E \circ i^* c = \int_c i_*(E(z)dz)$. Hence, $i_*(\mathcal{E}'_E)$ is the Eisenstein element inside the space of modular symbols corresponding to E. By Proposition 34 and 35, the boundary of $\pi_*(\mathcal{E}_E^0)$ is same as the boundary of $6i_*(\mathcal{E}'_E)$.

There is a non-degenerate bilinear pairing $S_2(\Gamma_0(pq)) \times \mathrm{H}_1(X_0(pq), \mathbb{R}) \to \mathbb{C}$ given by $(f, c) = \int_c f(z)dz$. Hence, the integrals of the holomorphic differentials over $\mathrm{H}_1(X_0(pq), \mathbb{Z})$ are not always zero. By Lemma 36, the integrals of every holomorphic differentials over $i_*(\mathcal{E}'_E)$ and $i_*(\pi_*(\mathcal{E}_E^0))$ are always zero. We deduce that

$$\mathcal{E}_E = i_*(\mathcal{E}'_E) = \frac{1}{6} i_*\pi_*(\mathcal{E}_E^0) = \frac{1}{6} \sum_{g \in \mathbb{P}(\mathbb{Z}/pq\mathbb{Z})} F_E(g)\xi(g).$$

for $E \in \mathbb{E}_{pq}$. Since $F_N(g) = \frac{1}{6} F_{E_N}(g)$, we obtain the theorem. $\square$

7.2. **The winding elements of level pq.** Recall the concept of the *winding element.*

Definition 37. [Winding elements] Let $\{0,\infty\}$ denote the projection of the path from 0 to ∞ in $\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})$ to $X_0(pq)(\mathbb{C})$. We have an isomorphism $\mathrm{H}_1(X_0(pq),\mathbb{Z})\otimes\mathbb{R} = \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}^0(X_0(pq),\Omega^1),\mathbb{C})$. Let $e_{pq}\in \mathrm{H}_1(X_0(pq),\mathbb{R})$ corresponds to the homomorphism $\omega \to -\int_0^\infty \omega$. The modular symbol e_{pq} is called the *winding element.*

The winding elements are the elements of the space of modular symbols whose annihilators define ideals of the Hecke algebras with the L-functions of the corresponding quotients of the Jacobian non-zero. In this paper, we found an explicit expression of the winding element. Let $e_{pq}\in \mathrm{H}_1(X_0(pq),\mathbb{Z})\otimes\mathbb{R}$ be the winding element. The following proposition help us to write down the winding element explicitly. Since $\sum_{x\in\partial(X_0(pq))} e_{\Gamma_0(pq))}(x)a_0(E[x])=0$, we write

$$\delta(E)=a_0(E)(\{\infty\}-\{0\})+qa_0(E[\frac{1}{p}])(\{\frac{1}{p}\}-\{0\})+pa_0(E[\frac{1}{q}])(\{\frac{1}{q}\}-\{0\}).$$

Lemma 38. *The constant Fourier coefficients of E_{pq} at cusps 0, $\frac{1}{p}$, $\frac{1}{q}$ and ∞ are $\frac{1-pq}{24pq}$, 0, 0 and $\frac{pq-1}{24}$ respectively.*

Proof. We first prove that the constant coefficient for the Fourier expansion of E_{pq} at the cusp $\frac{1}{p}$ is 0. As usual, the constant term of the Fourier expansion of E_{pq} at the cusp $\frac{1}{p}$ is the constant term at ∞ of $E_{pq}[\beta_0]$. Similarly, the constant term of the Fourier expansion of E_{pq} at the cusp $\frac{1}{q}$ is the constant term at ∞ of $E_{pq}[\gamma_0]$. Let Δ be the Ramanujan's cusp form of weight 12. We write $\frac{d}{dz}\log\Delta(\beta(z)) = 12\frac{d}{dz}\log(pz+1)+\frac{d}{dz}\log\Delta(z)$ for $\beta=\begin{pmatrix}1&0\\p&1\end{pmatrix}$. A simple calculation shows that

$$\Delta(\frac{pqz}{pz+1})=\Delta(\begin{pmatrix}q&0\\1&1\end{pmatrix}pz)=\Delta(\begin{pmatrix}q&-1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&q\end{pmatrix}pz)=$$

$$\Delta(\begin{pmatrix}q&-1\\1&0\end{pmatrix}(\frac{pz+1}{q}))=(\frac{pz+1}{q})^{12}\Delta(\frac{pz+1}{q}).$$

By taking logarithmic derivative, we deduce that

$$\frac{d}{dz}log\Delta\begin{pmatrix}q&-1\\1&0\end{pmatrix}(\frac{pz+1}{q})=12\frac{d}{dz}\log(pz+1)+\frac{d}{dz}\log\Delta(\frac{pz+1}{q}).$$

Since $E_{pq}(z)=\frac{1}{2\pi i}\frac{d}{dz}\log\frac{\Delta(pqz)}{\Delta(z)}$, the above calculation shows that the constant term of E_{pq} at the cusp $\frac{1}{p}$ is 0. Similarly, the constant term of E_{pq} at the cusp $\frac{1}{q}$ is 0. The constant term of E_{pq} is $\frac{pq-1}{24}$ at the cusp ∞ and $\frac{1-pq}{24pq}$ at 0. ☐

Using Lemma 36 and Lemma 38, we write

Corollary 39.

$$(1-pq)e_{pq}=\sum_{x\in(\mathbb{Z}/pq\mathbb{Z})^*}F_{pq}((1,x))\{0,\frac{1}{x}\}.$$

Remark 40. For the Eisenstein series $E_p \in E_2(\Gamma_0(p))$, $\frac{1}{p}$ represents the cusp ∞ and $\frac{1}{q}$ represents the cusp 0. We deduce that $a_0(E_p[\beta_0]) = \frac{p-1}{24}$ and $a_0(E_p[\gamma_0]) = \frac{1-p}{24p}$. For the other Eisenstein series $E_q \in E_2(\Gamma_0(q))$, $\frac{1}{q}$ represents the cusp ∞ and $\frac{1}{p}$ represents the cusp 0. We deduce that $a_0(E_q[\gamma_0]) = \frac{q-1}{24}$ and $a_0(E_q[\beta_0]) = \frac{1-q}{24q}$.

References

[1] A. Agashe, The Birch and Swinnerton-Dyer Conjecture for modular abelian varities of analytic rank zero, Ph.D. thesis, University of California at Berkeley (2000).

[2] D. Banerjee, *A note on the Eisenstein elements of prime square level*, Proceedings of the American Mathematical Society. (2014), no. 12, 3675–3686.

[3] F. Calegari and M. Emerton, *On the ramification of Hecke algebras at Eisenstein primes*, Invent. Math. **160** (2005), no. 1, 97–144.

[4] J. E. Cremona, Algorithms for modular elliptic curves, Cambridge University Press, Cambridge, second edition (1997), ISBN 0-521-59820-6.

[5] F. Diamond and J. Shurman, A first course in modular forms, Vol. 228 of *Graduate Texts in Mathematics*, Springer-Verlag, New York (2005), ISBN 0-387-23229-X.

[6] V. G. Drinfeld, *Two theorems on modular curves*, Funkcional. Anal. i Priložen. **7** (1973), no. 2, 83–84.

[7] S. Lang, Introduction to modular forms, Vol. 222 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin (1995), ISBN 3-540-07833-9. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.

[8] J. I. Manin, *Parabolic points and zeta functions of modular curves*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972) 19–66.

[9] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 33–186 (1978).

[10] ———, *On the arithmetic of special values of L functions*, Invent. Math. **55** (1979), no. 3, 207–240.

[11] L. Merel, *Intersections sur des courbes modulaires*, Manuscripta Math. **80** (1993), no. 3, 283–289.

[12] ———, *Homologie des courbes modulaires affines et paramétrisations modulaires*, in Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, 110–130, Int. Press, Cambridge, MA (1995).

[13] ———, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. **124** (1996), no. 1-3, 437–449.

[14] ———, *L'accouplement de Weil entre le sous-groupe de Shimura et le sous-groupe cuspidal de $J_0(p)$*, J. Reine Angew. Math. **477** (1996) 71–115.

[15] R. Miranda, Algebraic curves and Riemann surfaces, Vol. 5 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI (1995), ISBN 0-8218-0268-2.

[16] T. Miyake, Modular forms, Springer Monographs in Mathematics, Springer-Verlag, Berlin, english edition (2006), ISBN 978-3-540-29592-1; 3-540-29592-5. Translated from the 1976 Japanese original

by Yoshitaka Maeda.

[17] A. P. Ogg, *Hyperelliptic modular curves*, Bull. Soc. Math. France **102** (1974) 449–462.

[18] G. Stevens, Arithmetic on modular curves, Vol. 20 of *Progress in Mathematics*, Birkhäuser Boston, Inc., Boston, MA (1982), ISBN 3-7643-3088-0.

[19] ———, *The cuspidal group and special values of L-functions*, Trans. Amer. Math. Soc. **291** (1985), no. 2, 519–550.

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